INFLUENCE OF RESISTANCE FORCES ON EXISTENCE OF SUBHARMONIC OSCILLATIONS OF QUASILINEAR SYSTEMS

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The influence of the order of smallness of damping on the subharmonic oscillations of the order of $1/2$ in a system described by the Duffing equation is analyzed.

Let us consider a nonautonomous system with one degree of freedom

$$x^{\cdot\cdot} + \frac{1}{n^2} x = f(t) + \mu F(t, x, x^{\cdot}, \mu) \tag{1}$$

where n is a positive integer. The function $F(t, x, x^{\cdot}, \mu)$ is analytic in x and $x^{\cdot}$ in some domain of their variation, and in the small parameter μ for $0 \leqslant \mu < \mu_0$. Moreover $f(t)$ and $F(t, x, x^{\cdot}, \mu)$ are continuous, 2π-periodic functions of t.

The solution of the generating system is

$$x_0(t) = \varphi(t) + A_0 \cos\frac{t}{n} + nB_0 \sin\frac{t}{n} \tag{2}$$

Here the forced oscillations are 2π-periodic, while the natural oscillations are $2\pi n$-periodic.

We seek the subharmonic $2\pi n$-periodic oscillations of the order of $1/n$ of the initial system. The initial conditions are, as usual [1],

$$x(0) = \varphi(0) + A_0 + \beta, \quad x^{\cdot}(0) = \varphi^{\cdot}(0) + B_0 + \gamma \tag{3}$$

where β and γ are functions of μ and vanish when $\mu = 0$.

We write the solution of (1) in the form

$$x(t) = \varphi(t) + (A_0 + \beta)\cos\frac{t}{n} + n(B_0 + \gamma)\sin\frac{t}{n} + \sum_{m=1}^{\infty}\left[C_m(t) + \frac{\partial C_m(t)}{\partial A_0}\beta + \frac{\partial C_m(t)}{\partial B_0}\gamma + \ldots\right]\mu^m \tag{4}$$

while the amplitudes A_0 and B_0 are given by

$$C_1(2\pi n) = -n\int_0^{2\pi n} F(t, x_0, x_0^{\cdot}, 0)\sin\frac{t}{n}\,dt = 0 \tag{5}$$

$$C_1^{\cdot}(2\pi n) = \int_0^{2\pi n} F_0(t, x_0, x_0^{\cdot}, 0)\cos\frac{t}{n}\,dt = 0$$

The latter equations always have the solution $A_0 = 0$ and $B_0 = 0$ corresponding to the 2π-periodic solution [2]. They may also have nonzero solutions corresponding to the period of $2\pi n$.

A question arises whether the subharmonic oscillations will always be present in the given system.

Hale [3] succeded in obtaining the conditions of existence of the subharmonic oscillations for two types of equation. One of them represents a generalized Duffing equation

$$x^{\cdot\cdot} + \frac{1}{(2n+1)^2} x = \nu\cos t + \mu\sum_{s=0}^{r} c_s x^{2s+1} - \mu^k b x^{\cdot} \tag{6}$$

Let us denote by $[n/r]$ a smallest integer larger than n/r. Hale has shown that subharmonic oscillations exist in the given system if $k \geqslant [n/r]$. Thus the larger the value of n, i.e. the higher the order of the subharmonic oscillations, the weaker the damping sufficient to neutralize these oscillations. The estimate given above holds for odd order subharmonic oscillations.

We shall show how the influence of the resistance forces on the existence of subharmonic oscillations can be analyzed in specific cases, using the amplitude equations. We shall also consider the following Duffing equation as an example:

$$x^{\cdot\cdot} + n^{-2}x = \nu_0 \cos t + \lambda_0 \sin t + \mu\,(a_0 x + c_0 x^3) - \mu^k b_0 x^{\cdot} \tag{7}$$

where k is a positive integer. Let us make the transformation of time $t = n\tau$ and introduce the following notation ($n > 1$):

$$x' = dx\,/\,d\tau, \quad n^2 a_0 = a, \quad n^2 c_0 = c, \quad nb_0 = b$$
$$n^2\nu_0 = (1 - n^2)\nu, \quad n^2\lambda_0 = (1 - n^2)\lambda$$

We obtain

$$x'' + x = (1 - n^2)\nu \cos n\tau + (1 - n^2)\lambda \sin n\tau + \mu\,(ax + cx^3) - \mu^k bx' \tag{8}$$

The solution of the generating system is

$$x_0\,(\tau) = \nu \cos n\tau + \lambda \sin n\tau + A_0 \cos \tau + B_0 \sin \tau \tag{9}$$

As a result of the transformation $t = n\tau$ the natural oscillations now have a period of 2π, with the subharmonic oscillations of any order.

We consider the subharmonic oscillations of the order of $^1/_2$. First we construct the amplitude equations without damping ($b = 0$). We have

$$\begin{aligned} C_1\,(2\pi) &= -\pi B_0\,[a + {}^3/_2 c\,(\nu^2 + \lambda^2) + {}^3/_4 c\,(A_0^2 + B_0^2)] = 0 \\ C_1'\,(2\pi) &= \pi A_0\,[a + {}^3/_2 c\,(\nu^2 + \lambda^2) + {}^3/_4 c\,(A_0^2 + B_0^2)] = 0 \end{aligned} \tag{10}$$

The null solution of these equations is of no interest, since it corresponds to the solution of (8) π-periodic in τ. Equations (10) yield a single equation for the amplitudes A_0 and B_0

$$a + {}^3/_2 c\,(\nu^2 + \lambda^2) + {}^3/_4 c\,(A_0^2 + B_0^2) = 0 \tag{11}$$

Equation (11) can be written in the form

$$A_0^2 + B_0^2 = P \tag{12}$$

Thus we find that the solutions of the amplitude equations are real when either $P > 0$, or

$$\nu^2 + \lambda^2 < -\,{}^2/_3 a/c, \quad ac < 0 \tag{13}$$

The second equation is obtained from the terms of the order of μ^2, using the relation [4]

$$A_0 C_2\,(2\pi) + B_0 C_2'\,(2\pi) = 0 \tag{14}$$

Fairly complicated computations now give

$$2A_0B_0\,(A_0^2 - B_0^2)\,(\nu^2 - \lambda^2) - (A_0^4 - 6A_0^2B_0^2 + B_0^4)\nu\lambda = 0 \tag{15}$$

From (11) and (15) it follows that for $\nu \neq 0$ and $\lambda \neq 0$ neither A_0 nor B_0 is equal to zero. Let us devide both parts of (15) by B_0^4 and set $z = A_0\,/\,B_0$. Then we have

$$z^4 - 2lz^3 - 6z^2 + 2lz + 1 = 0, \quad l = (\nu^2 - \lambda^2)\,/\,\nu\lambda \tag{16}$$

It can easily be shown that when the parameter l is real, all roots of (16) are also real. This follows from the form of the curve depicting the left-hand side of Eq.(16) on a plane. This curve intersects the abscissa four times, since it passes through the points (0, 1) and (± 1, -4), and tends to $+\infty$ as $z \to \pm\infty$.

A general solution to the amplitude equations (11) and (15) cannot be obtained. For $\nu = 0$ or $\lambda = 0$, we have

$$A_0^2 = B_0^2 = -\,(\lambda^2 + {}^2/_3\, a/c), \qquad A_0 = B_0^2 = -\,(\nu^2 + {}^2/_3\, a/c) \tag{17}$$

for $\nu = \lambda$ we find

$$A_0^2 = -(2 \pm \sqrt{2})(\nu^2 + {}^1/_3\, a/c), \qquad B_0^2 = -(2 \mp \sqrt{2})(\nu^2 + {}^1/_3\, a/c) \tag{18}$$

The functional determinant of (11) and (15) does not vanish, consequently the amplitude equations have simple solutions. From this it follows that, for $n = 2$ and $b = 0$, the 2π-periodic solutions of (8) expand in the integral powers of μ.

Obviously the amplitude equations are not affected if damping of the order of μ^k for $k > 2$ is introduced into the system. It follows that the damping of the order indicated has no influence on the possibility of appearance of subharmonic oscillations considered here.

We now consider the subharmonic oscillations with damping of the order of μ^2. The first amplitude equation (11) remains unchanged, while the second one becomes

$$ {}^{33}/_{16}c^2\, [2A_0B_0\,(A_0^2 - B_0^2)(\nu^2 - \lambda^2) - (A_0^4 - 6A_0^2B_0^2 + B_0^4)\nu\lambda] - b\,(A_0^2 + B_0^2) = 0 \tag{19}$$

The latter can be made homogeneous by multiplying it by (12). Assuming that $\nu \neq 0$ and $\lambda \neq 0$ we obtain, after some transformations

$$z^4 - 2Mz^3 - Nz^2 + 2Mz + 1 = 0 \tag{20}$$

$$M = \frac{l}{1+Q}, \qquad N = 2\frac{3-Q}{1+Q}, \qquad Q = \frac{16}{33}\frac{b}{c^2}\frac{1}{\nu\lambda}\frac{1}{P} \tag{21}$$

The curve representing the left-hand side of Eq.(20) passes through the points (0, 1) and $(\pm 1,\ 2-N)$. Consequently we can assert that Eq.(20) has real solutions if at least $2 - N \leqslant 0$. Thus $Q \leqslant 1$ represents the sufficient condition of existence of real solutions. It can be shown that at sufficiently large Q the subharmonic oscillations will be absent.

If either ν or λ becomes zero, e. g. $\lambda = 0$, then Eq.(20) can be written as

$$R\,(z^2 + 1)^2 - 2z\,(z^2 - 1) = 0, \qquad R = \frac{16}{33}\frac{b}{c^2}\frac{1}{\nu^2 P} \tag{22}$$

When $R = 0$, its solution is (17). Obviously real solutions of (22) also exist on some segment $0 \leqslant R \leqslant R_0 < 1$.

Thus when damping is of the order of μ^2 subharmonic oscillations of the order of $^1/_2$ can be obtained, but only over a more narrow range of the coefficients of the Duffing equation than that spanned by the oscillations when damping is absent.

Finally we discuss the problem of existence of subharmonic oscillations of the order of $^1/_2$ in the Duffing system for $k = 1$. In this case the amplitude equations are

$$C_1(2\pi) = -\pi\,\{B_0\,[a + {}^3/_2\, c\,(\nu^2 + \lambda^2) + {}^3/_4\, c\,(A_0^2 + B_0^2)] - bA_0\} = 0 \tag{23}$$

$$C_1'(2\pi) = \pi\,\{A_0\,[a + {}^3/_2\, c\,(\nu^2 + \lambda^2) + {}^3/_4\, c\,(A_0^2 + B_0^2)] + bB_0\} = 0$$

These equations have a unique real solution $A_0 = 0,\ B_0 = 0$. It follows that when damping of the order of μ is introduced, subharmonic oscillations of the order of $^1/_2$ are impossible in the Duffing system.

Thus the method of analyzing the amplitude equations can be employed in solving specific cases of the problem of conditions of existence of the subharmonic oscillations.

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